404. Proposed by S. LEFSCHETZ, Ph. D., University of Nebraska.

Let ABC be a triangle, M a point on BC, BE the intersection of the perpendicular to AM in M with AB and AC, F the other intersection of the circle circumscribed to ABC with the circle through M and A orthogonal to it in A. Prove that the points C, D, E, F are on a circle, and find the envelope of the latter when M describes BC.

No solution of this problem has been received.

405. Proposed by C. N. SCHMALL, New York City.

A plane cuts a constant volume from a given right cone. Prove that the minor axis of the section has a constant length.

Solution by the PROPOSER.

Let C be the vertex of the cone, AB the major axis of any one of the sections made by the plane. Let F be the point of contact of the inscribed circle of triangle ABC with the side AB. Then F is a focus of the section. Draw CD perpendicular to AB. Let V denote the volume cut off. Then, if α, β, denote the semi-axes of the section, we have

$$V=\tfrac{1}{3}CD.\,(\text{area of section})=\tfrac{1}{3}CD.\pi\,\alpha\,\beta=\tfrac{1}{3}\,\pi.CD.\alpha\,\beta$$
$$=\tfrac{1}{6}\,\pi.CD.AB.\beta=\tfrac{1}{6}\,\pi\,ab\sin C.\beta\ldots(1),$$

(where a, b, are the sides AC, BC, of the triangle ABC). But, $\beta^2=AF.FB=(s-a)(s-b)=ab\sin^2\tfrac{1}{2}C$. $\therefore ab=\beta^2\cos^2\tfrac{1}{2}C$.

Substituting this in (1), we get, $V=\tfrac{1}{3}\,\pi\,\beta^3.\tan\tfrac{1}{2}C$.

Now, since C and V are constant, therefore β, or the minor axis (2β) is also constant. It is now also evident, by (1), that the area of triangle ABC is constant.

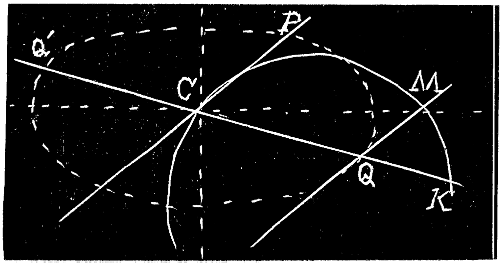
406. Proposed by DR. R. K. MORLEY, University of Illinois.

Given the lengths of a pair of conjugate diameters of an ellipse and the angle between them; to construct (with ruler and compass) the axes of the ellipse, *i. e.*, find their lengths and the angles they make with the given diameters.

I. Solution by A. M. HARDING, University of Arkansas.

Let PP' and QQ' be the conjugate diameters, and C the center. Produce CQ to K, so that $CQ.QK=CP$.

Draw a line through Q parallel to CP. Construct a circle with its center on this line and passing through C and K. Let the circle cut the line through Q at M and N. Then the axes of the ellipse will lie along CM and CN.



Proof. MN is tangent to the conic.

Now it can be easily shown that if a tangent meets any two lines in such a way that the product of its segments equals the square on the parallel semi-diameter, then the two lines are parallel to a pair of conjugate diameters. In the above construction, $NQ.QM=CQ.QK=CP^2$.

Therefore, CM and CN are the axes (since they are at right angles). Now let r=side of a square whose area is four times the triangle QCP; then $r^2=2ab$. Let s=hypotenuse of a right triangle whose legs are CP and CQ; then $s^2=CP^2+CQ^2=a^2+b^2$. Now construct $m=\sqrt{(s^2+r^2)}$ and $n=\sqrt{(s^2-r^2)}$; then $m=a+b$, $n=a-b$; hence, $a=\dfrac{m+n}{2}$, $b=\dfrac{m-n}{2}$.

II. **Solution by H. C. FEEMSTER, York College, York, Nebraska.**

Let $a'a$ and $b'b$ be the conjugate diameters, making the angle aOb. Through a draw a parallel to Ob, this will be a tangent to the ellipse. Extend Oa to P, so that Ob is a mean proportional between Oa and OP. Draw a perpendicular bisector of Op intersecting the tangent through a at C. With C as a center draw a circle through P and O, cutting aC in A and B. Draw OA and OB; these will be the semi-axes of the ellipse, for $Aa\times aB=Oa\times aP=Ob^2$. OA and OB are conjugate diameters, and are perpendicular to each other. And these make the required angles with the given diameters. See Salmon's *Conic Sections*, pages 172 and 173.

Also solved by C. N. Schmall.

CALCULUS.

325. **Proposed by E. B. ESCOTT, Ann Arbor, Michigan.**

Integrate $\displaystyle\int\frac{4x^6-a^3}{\sqrt{(x^6-a^3)}}dx.$

Solution by MARY E. WILSON, Olivet College, Olivet, Michigan.

Let $x^2=z$. Then $dx=\frac{1}{2}\dfrac{dz}{\sqrt{z}}$. Hence, $\displaystyle\int\frac{4x^6-a^3}{\sqrt{(x^6-a^3)}}=\int\frac{4z^3-a^3}{2\sqrt{(z^4-za^3)}}dz$

$$=\sqrt{(z^4-a^3z)}+C=x\sqrt{(x^6-a^3)}+C.$$

Also solved by G. W. Hartwell, and the Proposer. The Proposer multiplied both numerator and denominator by x, after which the integral is seen directly.